Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Note on the Differential Invariants of Goursat and Painlevé.

BY E. O. LOVETT.

In a memoir on linear homogeneous differential equations published in the last number of the Journal de l'École Polytechnique, second series, fourth cahier, M. A. Boulanger constructs a class of differential invariants first studied by MM. Goursat and Painlevé for ternary and quaternary finite linear groups. In deriving the forms of these invariants for quaternary groups, the author makes use of the process of elimination employed by Sophus Lie* for the construction of differential invariants when the finite equations of a group are given. The method of procedure followed by M. Boulanger leads readily to a class of differential invariants under the n-ary finite linear group, as in fact he himself predicted.

Consider a finite group of linear substitutions in n non-homogeneous variables u_i and the n corresponding fundamental invariant functions

$$x_i = f_i(u_1, u_2, \ldots, u_n), \qquad i = 1, 2, \ldots, n. \tag{1}$$

Let the defining equations of the group be

$$u_i = \frac{a_{i,1} u_1' + a_{i,2} u_2' + \ldots + a_{i,n} u_n' + a_{i,n+1}}{a_{0,1} u_1' + a_{0,2} u_2' + \ldots + a_{0,n} u_n' + a_{0,n+1}}, \qquad i = 1, \ldots, n, \tag{2}$$

which, for convenience, are written,

$$\left.\begin{aligned}
&a_{0,1} u_1 u_1' + a_{0,2} u_1 u_2' + \ldots + a_{0,n} u_1 u_n' + a_{0,n+1} u_1 \\
&\qquad = a_{1,1} u_1' + a_{1,2} u_2' + \ldots + a_{1,n} u_n' + a_{1,n+1}, \\
&a_{0,1} u_2 u_1' + a_{0,2} u_2 u_2' + \ldots + a_{0,n} u_2 u_n' + a_{0,n+1} u_2 \\
&\qquad = a_{2,1} u_1' + a_{2,2} u_2' + \ldots + a_{2,n} u_n' + a_{2,n+1}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_{0,1} u_n u_1' + a_{0,2} u_n u_2' + \ldots + a_{0,n} u_n u_n' + a_{0,n+1} u_n \\
&\qquad = a_{n,1} u_1' + a_{n,2} u_2' + \ldots + a_{n,n} u_n' + a_{n,n+1}.
\end{aligned}\right\} \tag{3}$$

---

* See Lie, "Theorie der Transformationsgruppen," Unter Mitwirkung von F. Engel, volume I, chapters XIII and XXV.

By differentiating the first equation of this system (3) successively with regard to $x_1, x_2, \ldots\ldots, x_n$, we have

$$\left.\begin{aligned}
&(a_{0,1} u_1' + u_{0,2} u_2' + \ldots\ldots + a_{0,n} u_n' + a_{0,n+1}) \frac{\partial u_1}{\partial x_1} \\
&\qquad\qquad = (a_{1,1} - a_{0,1} u_1) \frac{\partial u_1'}{\partial x_1} + \ldots\ldots + (a_{1,n} - a_{0,n}) \frac{\partial u_n'}{\partial x_1}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\
&(a_{0,1} u_1' + a_{0,2} u_2' + \ldots\ldots + a_{0,n} u_n' + a_{0,n+1}) \frac{\partial u_1}{\partial x_n} \\
&\qquad\qquad = (a_{1,1} - a_{0,1} u_1) \frac{\partial u_1'}{\partial x_n} + \ldots\ldots + (a_{1,n} - a_{0,n}) \frac{\partial u_n'}{\partial x_n}.
\end{aligned}\right\} \quad (4)$$

From this system (4) we deduce by solution that

$$\left.\begin{aligned}
a_{1,1} &= a_{0,1} u_1 + \frac{D(u_1, u_2', \ldots\ldots, u_n')}{D(u_1, u_2, \ldots\ldots, u_n)} (a_{0,1} u_1' + \ldots\ldots + a_{0,n} u_n' + a_{0,n+1}), \\
a_{1,2} &= a_{0,2} u_1 + \frac{D(u_1', u_1, u_3', \ldots\ldots, u_n')}{D(u_1, u_2, \ldots\ldots, u_n)} (a_{0,1} u_1' + \ldots\ldots + a_{0,n} u_n' + a_{0,n+1}), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{1,n} &= a_{0,n} u_1 + \frac{D(u_1', u_2', \ldots\ldots, u_{n-1}', u_1)}{D(u_1, u_2, \ldots\ldots, u_n)} (a_{0,1} u_1' + \ldots\ldots + a_{0,n} u_n' + a_{0,n+1}),
\end{aligned}\right\} \quad (5)$$

where $D(u_1', u_2', \ldots\ldots, u_n')$ represents the functional determinant of $u_1', \ldots\ldots, u_n'$ with regard to $x_1, \ldots\ldots, x_n$.

Differentiating this same first equation of the system (3), first twice with regard to x_1, and then with regard to x_1 and x_2, and substituting the above values of $a_{1,1}, a_{1,2}, \ldots\ldots, a_{1,n}$ in the results, we obtain

$$\left.\begin{aligned}
&2\left(a_{0,1} \frac{\partial u_1'}{\partial x_1} + \ldots\ldots, + a_{0,n} \frac{\partial u_n'}{\partial x_1}\right) \frac{\partial u_1}{\partial x_1} \\
&\quad = \frac{a_{0,1} u_1' + \ldots\ldots + a_{0,n} u_n' + a_{0,n+1}}{D(u_1, u_2, \ldots\ldots, u_n)} \left\{ - \frac{\partial^2 u_1}{\partial x_1^2} D(u_1', \ldots\ldots, u_n') \right. \\
&\quad \left. + \frac{\partial^2 u_1'}{\partial x_1^2} D(u_1, u_2', \ldots\ldots, u_n') + \ldots\ldots + \frac{\partial^2 u_n'}{\partial x_1^2} D(u_1', u_2', \ldots\ldots, u_{n-1}', u_1) \right\}, \\
&\left(a_{0,1} \frac{\partial u_1'}{\partial x_1} + \ldots\ldots + a_{0,n} \frac{\partial u_n'}{\partial x_1}\right) \frac{\partial u_1}{\partial x_2} + \left(a_{0,1} \frac{\partial u_1'}{\partial x_2} + \ldots\ldots + a_{0,n} \frac{\partial u_n'}{\partial x_2}\right) \frac{\partial u_1}{\partial x_1} \\
&\quad = \frac{a_{0,1} u_1' + \ldots\ldots + a_{0,n} u_n' + a_{0,n+1}}{D(u_1, u_2, \ldots\ldots, u_n)} \left\{ - \frac{\partial^2 u_1'}{\partial x_1 \partial x_2} D(u_1', \ldots\ldots, u_n') \right. \\
&\quad \left. + \frac{\partial^2 u_1}{\partial x_1 \partial x_2} D(u_1, u_2', \ldots\ldots, u_n') + \ldots + \frac{\partial^2 u_n}{\partial x_1 \partial x_2} D(u_1', u_2', \ldots\ldots, u_{n-1}', u_1) \right\}.
\end{aligned}\right\} \quad (6)$$

For convenience put

$$(a_{0,1}u_1' + \dots\dots + a_{0,n}u_n' + a_{0,n+1})\,\xi_i$$
$$= \left(a_{0,1}\frac{\partial u_1'}{\partial x_i} + \dots\dots + a_{0,n}\frac{\partial u_n'}{\partial x_i}\right) D(u_1', u_2', \dots\dots, u_n').$$

Then the above equations (6) become

$$\left.\begin{array}{l}
2\xi_1\dfrac{\partial u_1}{\partial x_1} + \begin{vmatrix} \dfrac{\partial^2 u_1}{\partial x_1^2} & \dfrac{\partial^2 u_1'}{\partial x_1^2} & \dfrac{\partial^2 u_2'}{\partial x_1^2} & \dots\dots & \dfrac{\partial^2 u_n'}{\partial x_1^2} \\ \dfrac{\partial u_1}{\partial x} & \dfrac{\partial u_1'}{\partial x_1} & \dfrac{\partial u_2'}{\partial x_1} & \dots\dots & \dfrac{\partial u_n'}{\partial x_1} \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{\partial u_1}{\partial x_n} & \dfrac{\partial u_1'}{\partial x_n} & \dfrac{\partial u_n'}{\partial x_n} & \dots\dots & \dfrac{\partial u_n'}{\partial x_n} \end{vmatrix} = 0, \\[4ex]
\xi_1\dfrac{\partial u_1}{\partial x_2} + \xi_2\dfrac{\partial u_1}{\partial x_1} + \begin{vmatrix} \dfrac{\partial^2 u_1}{\partial x_1\partial x_2} & \dfrac{\partial^2 u_1'}{\partial x_1\partial x_2} & \dfrac{\partial^2 u_2'}{\partial x_1\partial x_2} & \dots\dots & \dfrac{\partial^2 u_n'}{\partial x_1\partial x_2} \\ \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1'}{\partial x_1} & \dfrac{\partial u_2'}{\partial x_1} & \dots\dots & \dfrac{\partial u_n'}{\partial x_1} \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{\partial u_1}{\partial x_n} & \dfrac{\partial u_1'}{\partial x_n} & \dfrac{\partial u_2'}{\partial x_n} & \dots\dots & \dfrac{\partial u_n'}{\partial x_n} \end{vmatrix} = 0;
\end{array}\right\} \qquad (7)$$

in the same way, by differentiating the first of equations (3) twice with regard to the other variables, we obtain $\frac{1}{2}\{n(n+1) - 4\}$ other equations,

$$2\xi_2\frac{\partial u_1}{\partial x_2} + \begin{vmatrix} \dfrac{\partial^2 u_1}{\partial x_2^2} & \dfrac{\partial^2 u_1'}{\partial x_2^2} & \dfrac{\partial^2 u_2'}{\partial x_2^2} & \dots\dots & \dfrac{\partial^2 u_n'}{\partial x_2^2} \\ \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1'}{\partial x_1} & \dfrac{\partial u_2'}{\partial x_1} & \dots\dots & \dfrac{\partial u_n'}{\partial x_1} \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{\partial u_1}{\partial x_n} & \dfrac{\partial u_1'}{\partial x_n} & \dfrac{\partial u_2'}{\partial x_n} & \dots\dots & \dfrac{\partial u_n'}{\partial x_n} \end{vmatrix} = 0, \dots\dots; \qquad (8)$$

and by treating the remaining equations of (3) one by one in like manner, we have, all told, $\frac{1}{2}n^2(n+1)$ equations from which the n functions $\xi_1, \xi_2, \dots\dots, \xi_n$ may be eliminated. The resulting $\frac{1}{2}n(n-1)(n+2)$ equations define as many differential invariants which come to light as follows.

The functions ξ_1 and ξ_2 can be eliminated from the three equations written above by multiplying the first by $\left(\frac{\partial u_1}{\partial x_2}\right)^2$, the second by $-\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}$, the third by $\left(\frac{\partial u_1}{\partial x_1}\right)^2$, and adding member to member. We thus have

$$\begin{aligned}
&\left\{\frac{\partial^2 u_1}{\partial x_1^2}\left(\frac{\partial u_1}{\partial x_2}\right)^2 - 2\frac{\partial u_1}{\partial x_1}\,\frac{\partial u_1}{\partial x_2}\,\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_1}{\partial x_2^2}\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\} D(u_1', u_2', \ldots, u_n') \\
&\quad - \frac{\partial u_1}{\partial x_1}\left(\frac{\partial u_1}{\partial x_2^2}\right)\left\{\left|\frac{\partial^2 u_1'}{\partial x_1^2}, \frac{\partial u_2'}{\partial x_2}, \frac{\partial u_3'}{\partial x_3}, \ldots, \frac{\partial u_n'}{\partial x_n}\right| - 2\left|\frac{\partial^2 u_1'}{\partial x_1 \partial x_2}, \frac{\partial u_2'}{\partial x_1}, \frac{\partial u_3'}{\partial x_3}, \ldots, \frac{\partial u_n'}{\partial x_n}\right|\right\} \\
&\quad + \left(\frac{\partial u_1}{\partial x_2}\right)^3 \left|\frac{\partial^2 u_1'}{\partial x_1^2}, \frac{\partial u_2'}{\partial x_1}, \frac{\partial u_3'}{\partial x_3}, \ldots, \frac{\partial u_n'}{\partial x_n}\right| \\
&\quad - \left(\frac{\partial u_1}{\partial x_1}\right)^3 \left|\frac{\partial^2 u_1'}{\partial x_2^2}, \frac{\partial u_2'}{\partial x_2}, \frac{\partial u_3'}{\partial x_3}, \ldots, \frac{\partial u_n'}{\partial x_n}\right| + \ldots = 0,
\end{aligned} \tag{9}$$

The ratios of any two of the coefficients of this equation (in which coefficients only the derivatives of u_1', u_2', ..., u_n' enter) are not changed by a linear substitution.

By forming the equations analogous to (9), we obtain the following system of differential invariants, omitting the factor $1/D(u_1', \ldots, u_n')$ and writing

$$\left|\frac{\partial^2 u_1'}{\partial x_h \partial x_k}, \frac{\partial u_2'}{\partial x_l}, \frac{\partial u_3'}{\partial x_m}, \ldots, \frac{\partial u_n'}{\partial x_\mu}\right| \equiv \{x_h x_k, x_l, x_m, \ldots, x_\mu\},$$

namely, $n(n-1)$ of the form

$$\{x_i x_i, x_i, x_j, \ldots, x_\mu\}, \qquad 10)$$

where

$$j \mp \mu \mp i; \qquad i = 1, 2, \ldots, n,$$

and x_j, ..., x_μ is a cyclical combination of the remaining $n-1$ taken $n-2$ at a time; $\frac{1}{2}n(n-1)(n-2)$ of the form

$$\{x_i x_j, x_i, x_j, x_k, \ldots, x_\mu\}, \qquad 11)$$

where

$$j \mp k \mp \mu \mp i; \quad i = 1, \ldots, n; \; j = 1, \ldots, n;$$

and x_k, ..., x_μ is a cyclical combination of the remaining; and $n(n-1)$ of the form

$$\{x_{ii}, x_j, x_{j+1}, \ldots, x_{i-1}\} - 2\{x_i x_j, x_i, x_{j+1}, \ldots, x_{i-1}\}, \tag{12}$$
$$i = 1, \ldots, n; \; j = 1, \ldots, n.$$

This system of invariants (10), (11), (12) may be regarded as an extension of Schwarz's invariant

$$\frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2$$

to the case of n variables.

For $n=2$, the system (10), (11), (12) gives the system set up by M. Goursat (Comptes Rendus, vol. 104, p. 1362),

$$\frac{\partial v}{\partial y}\,\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\,\frac{\partial^2 v}{\partial y^2}=A\Delta,\qquad \frac{\partial u}{\partial x}\,\frac{\partial^2 v}{\partial x^2}-\frac{\partial v}{\partial x}\,\frac{\partial^2 u}{\partial x^2}=D\Delta,$$

$$\frac{\partial u}{\partial x}\,\frac{\partial^2 v}{\partial y^2}-\frac{\partial v}{\partial x}\,\frac{\partial^2 u}{\partial y^2}+2\left(\frac{\partial u}{\partial y}\,\frac{\partial^2 v}{\partial x\,\partial y}-\frac{\partial v}{\partial y}\,\frac{\partial^2 u}{\partial x\,\partial y}\right)=B\Delta,$$

$$\frac{\partial v}{\partial y}\,\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial y}\,\frac{\partial^2 v}{\partial x^2}+2\left(\frac{\partial v}{\partial x}\,\frac{\partial^2 u}{\partial x\,\partial y}-\frac{\partial u}{\partial x}\,\frac{\partial^2 v}{\partial x\,\partial y}\right)=C\Delta,$$

where

$$\Delta=\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\,\frac{\partial u}{\partial y}\,.$$

For $n=3$, we have the system constructed by Painlevé (Comptes Rendus, vol. 104, p. 1500),

$$\begin{array}{lll} |T_{xy},\ U_x,\ V_y|, & |T_{yz},\ U_y,\ V_z|, & |T_{zx},\ U_z,\ V_x|, \\ |T_{xx},\ U_x,\ V_y|, & |T_{xx},\ U_x,\ V_z|, & |T_{yy},\ U_y,\ V_z|, \\ |T_{yy},\ U_y,\ V_x|, & |T_{zz},\ U_z,\ V_x|, & |T_{zz},\ U_z,\ V_y|. \end{array}$$

$$\begin{array}{ll} |T_{xx},\ U_y,\ V_z|-2|T_{xy},\ U_x,\ V_z|, & |T_{xx},\ U_y,\ V_z|-2|T_{xz},\ U_y,\ V_x|, \\ |T_{yy},\ U_z,\ V_x|-2|T_{xy},\ U_z,\ V_y|, & |T_{yy},\ U_z,\ V_y|-2|T_{xz},\ U_y,\ V_x|, \\ |T_{xz},\ U_x,\ V_y|-2|T_{zx},\ U_z,\ V_y|, & |T_{zz},\ U_x,\ V_y|-2|T_{xu},\ U_x,\ V_z|, \end{array}$$

where the factor $1/D(T,\ U,\ V)$ has been omitted.

PRINCETON, NEW JERSEY, *May* 16, 1899.